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Cosmological constraints on $f(R)$

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Abstract. Modifications to general relativity have been suggested as viable alternatives to dark energy, introduced to explain the accelerated expansion of the Universe. We perform a Bayesian analysis on modified gravity models using current cosmological observations. We investigate the evolution both of the background universe and density perturbations. While the cosmic expansion can be recast using an effective equation-of-state $w_{\rm eff}(a)$, the evolution of linear perturbations is studied by the introduction of two parametric functions: the ratio of the two metric potentials and the ratio of an effective gravitational constant to the Newtonian constant in the Poisson equation. With the use of large-scale structure, cosmic microwave background and supernovae data we are be able to impose constraints on any $f(R)$ model, in particular we consider a variant of the Starobinsky model $f(R) = R - \lambda\, R_c \left[1 - \left(1 + \alpha \frac{R}{R_c}\right)^{-n}\right]$ parameterised in terms of λ, α and n. We find that, for $n = 2$, current cosmological observations limit $\lambda = 15.6 \pm 1.3$, $\alpha > 0.4$, and the present value of the field-amplitude $0 < F_0 - 1 < 0.998$, and its effective equation-of-state today $-1 < w_{\rm eff,0} < -0.998$, at 95% C.L. In addition to parameter estimation, we compare the family of models using the Bayesian model selection. We find that our $f(R)$ model fits slightly better to current data compared to the standard ΛCDM model. The approach here performed can be extended to any $f(R)$ model in order to test possible deviations from the standard cosmological model.

Keywords: cosmological parameters – cosmology: observations – cosmology: theory – cosmic background radiation – large-scale structure of Universe – modified gravity

Contents

1 Introduction

Since the discovery of the accelerated expansion of the Universe, many efforts have been made to understand the cause of this remarkable phenomenon [48, 52]. The leading model proposed so far assumes the validity of general relativity (GR) with the introduction of *dark matter* and *dark energy components*, which together account for 96% of the total energy density of the Universe. While dark matter plays a key role in structure formation, dark energy is introduced to explain the late-time acceleration of the Universe. The simplest form of the dark energy component, described by a perfect fluid with a constant equation-of-state $w(z) = -1$, the well known cosmological constant Λ, leads to the standard ΛCDM model. Although ΛCDM provides a good fit to current data, it still faces some theoretical challenges, such as the coincidence and fine-tuning problems [13, 46, 47]. Also, current experiments, with the use of model-independent techniques for reconstructing the properties of dark energy, support a mild time-dependent evolution of $w(z)$ [3, 28, 36, 62, 69]. Hence, to attempt to deal with these difficulties, alternative proposals to the conventional cosmological constant term have emerged. Several options involve the introduction of new exotic fluids to the energy-momentum tensor, such as quintessence and K-essence, amongst many other scenarios [9, 16, 50, 63, 70]. Another popular route is provided by non-minimally coupling scalar-fields to gravity and to matter [5, 17, 29]. Besides the dark sector, there exists a variety of models where the present expansion is realised due to modifications of the laws of gravity (MG) on cosmological scales. Some of them introduce non-linear terms to the standard Einstein-Hilbert action, like $f(R)$ theories [8, 31, 44, 54, 58], or higher dimensional braneworld models [21, 39]. A further interesting possibility is to take the view that anisotropies might be responsible for the observed acceleration [2, 43]. Other alternatives and combinations are also considered as good candidates [for a review see: 18, 20, 23, and references therein].

As a consequence of the introduction of different models, a fundamental question arises regarding how to distinguish amongst these possibilities. The cosmic acceleration, produced by any of these proposals, might affect both the expansion history and the growth rate of large-scale structure in the Universe. Hence, a natural search for departures from the ΛCDM model is to exploit present and future cosmological observations. At the background level, we assume the evolution is correctly described by the spatially-flat Friedmann-Robertson-Walker (FRW) metric with a time-dependent scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)dx^i dx_i. \tag{1.1}$$

The cosmic evolution driven by a modified gravity model may be described in terms of an effective equation-of-state $w_{\rm eff}(z)$, which is fitted with luminosity distance constraints, e.g. from high-redshift supernovae data. However, the extra degrees of freedom in extended GR models result in more freedom to reproduce any desired background evolution. That is, given the Hubble rate $H = H(a)$, one can identify a complete family of models such that the Friedmann equation is satisfied, and hence at the background level, suitable MG theories might be indistinguishable from ΛCDM or dark energy theories [45, 56]. Then, our analysis is mainly focussed on the cosmological perturbations. In this context, two scalar field potentials, $\Psi(t, \boldsymbol{x})$ and $\Phi(t, \boldsymbol{x})$, specify the evolution of linear perturbations around a flat FRW background:

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 - 2\Phi)\gamma_{ij} dx^i dx^j. \tag{1.2}$$

Standard ΛCDM and models with minimally coupled scalar fields are based on the assumption that the Newtonian potentials Φ and Ψ, satisfy the relation $\Phi = \Psi$. Nevertheless, modified gravity models usually predict the existence of an effective anisotropic stress, so the two metric potentials are no longer necessarily the same. Thus, we may differentiate a dark energy model from a particular modified gravity theory via the relationship between the two metric potentials Φ, Ψ and the density perturbation δ_m [33].

For the purpose of detecting possible departures from general relativity, we incorporate time and scale-dependent functions, $\mu(a, k)$ and $\gamma(a, k)$, into the Poisson and anisotropy equations in standard GR. This approach has been employed before in the search for departures from ΛCDM, see for instance [7, 27, 40, 57, 68]. Then, for a generic MG theory, the linearised Einstein-like equations have the following form

$$\frac{k^2}{a^2}\Psi = -\frac{\kappa^2}{2}\mu(a, k)\,\delta\rho_m, \tag{1.3}$$

$$\Phi = \gamma(a, k)\,\Psi, \tag{1.4}$$

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi = 0. \tag{1.5}$$

The *screened mass function* μ is interpreted as the ratio of an effective gravitational constant relative to the Newtonian constant, $\mu \equiv G_{\rm eff}(a, k)/G_{\rm N}$. The other relevant function, the gravitational *slip parameter* γ, defined as the ratio of the spatial perturbation to the time-time perturbation of the metric $\gamma \equiv \Phi/\Psi$, is seen as an effective anisotropic stress. We observe that modifications of GR, for which $\mu(a, k) = \gamma(a, k) = 1$, affect, through the Newtonian potentials, the growth of matter density perturbations δ_m, as shown in equation (1.5). Thus, current or future surveys, may allow us to distinguish modified gravity models from general relativity with a dark energy component.

In general, the action for modified gravity $f(R)$ models may be written as

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\, f(R) + S_{\rm M}(g_{\mu\nu}, \psi_{\rm M}), \tag{1.6}$$

where $\kappa^2 \equiv 8\pi G$, g is the determinant of the metric $g_{\mu\nu}$, $f(R)$ is some arbitrary function of the Ricci scalar R, although in the most general case it may also contain an scalar field dependency $f(R,\phi)$; the matter action $S_{\rm M}$ depends on $g_{\mu\nu}$ and matter fields $\psi_{\rm M}$. Notice that the action (1.6) contains, in particular, the standard Einstein-Hilbert action $f(R) = R - 2\Lambda$. Thus we consider the standard metric formalism by performing the variation of the action with respect to the metric tensor $g_{\mu\nu}$, bearing in mind that in the Palatini formalism different field equations may arise for a Lagrangian density non-linear in R [26, 65]. Varying the action with respect to $g_{\mu\nu}$, the following field equations are obtained [23]

$$FG_{\mu\nu} - \frac{1}{2}(f - RF)g_{\mu\nu} - F_{,\mu;\nu} + \Box F g_{\mu\nu} = \kappa^2 T^{\rm (M)}_{\mu\nu}, \tag{1.7}$$

where subscripts '$_{,X}$' stand for partial derivative with respect to the variable X, e.g, $F(R) \equiv \partial f/\partial R = f_{,R}$ and likewise $F_{,R} = f_{,RR} = \partial^2 f/\partial R^2$. Also $\dot{} \equiv d/dt$ and $' \equiv d/d\ln a$ below; $G_{\mu\nu}$ is the Einstein tensor.

In this work we particularly focus on a version of the Starobinsky model, and then use current SNe, LSS and CMB data to constrain the parameter-space. Finally, because the addition of parameters to the standard model may lead to an arbitrary accurate fit, we consider the Bayesian evidence as a quantitative implementation of Occam's razor. In this way, we obtain the model preferred by current observations.

The outline of the paper is as follows. In Section 2, we discuss the background evolution and scalar perturbations for a modified gravity theory, in particular $f(R)$ models. Section 3 describes the parameter estimation and model selection analysis. We then specify observable quantities used to constrain the parameter-space through current data sets. The constraints on the parameters used to describe the modified gravity models, along with Bayesian evidence values, are discussed in Section 4. We present our conclusions in Section 5.

2 $f(R)$ Gravity

The simplest family of MG models that gives rise to the acceleration of the universal expansion are obtained by replacing the Ricci scalar R in the usual Einstein-Hilbert Lagrangian by a non-linear function of R. Since modifications of gravity are more apparent at low redshift, we henceforth ignore the radiation component due to its relative unimportance for structure formation at late time. We thus base our analysis on non-relativistic matter with background energy density ρ_m and negligible pressure $P_m = 0$.

2.1 Background evolution

For the background evolution, the metric (1.1) leads to the Ricci scalar given by

$$R = 6(2H^2 + \dot{H}). \tag{2.1}$$

The modified Friedmann equation then becomes:

$$3FH^2 = (FR - f)/2 - 3H\dot{F} + \kappa^2\rho_m. \tag{2.2}$$

To find solutions for H and R, we follow [31, 41] and introduce new variables, which vanish in the high-redshift limit where $f(R)$ modifications are negligible:

$$y_H \equiv \frac{H^2}{R_c} - a^{-3}, \qquad y_R \equiv \frac{R}{R_c} - 3a^{-3}, \tag{2.3}$$

with R_c given in terms of the average matter-density today $\rho_{m,0}$, by $R_c = \kappa^2\rho_{m,0}/3$. Thus, equations (2.1) and (2.2) are expressed as a set of ordinary differential equations

$$y'_H = \frac{1}{3}y_R - 4y_H, \tag{2.4}$$

$$y'_R = 9a^{-3} - \frac{1}{y_H + a^{-3}}\frac{1}{R_c f_{,RR}}\left[y_H - (f_{,R} - 1)\left(\frac{1}{6}y_R - y_H - \frac{1}{2}a^{-3}\right) + \frac{1}{6}\frac{f - R}{R_c}\right]. \tag{2.5}$$

It may be shown that the expansion history generated by a $f(R)$ model is identical to that of a standard dark-energy model with an effective equation-of-state:

$$1 + w_{\text{eff}} = -\frac{1}{3}\frac{y'_H}{y_H}. \tag{2.6}$$

2.2 Scalar perturbations

The evolution of the scalaron field F, is determined from the trace of Equation (1.7)

$$3\,\Box F(R) + F(R)R - 2f(R) = -\kappa^2\rho_m. \tag{2.7}$$

This field equation can be written as a Poisson equation $\Box F(R) = \partial V_{eff}/\partial f_R$, with an effective potential

$$\frac{\partial V_{eff}}{\partial F} = \frac{1}{3}\left[2f(R) - F(R)R - \kappa^2\rho_m\right], \tag{2.8}$$

which presents an extremum value located at $2f(R) - F(R)R = \kappa^2\rho_m$.
In the high-density region, where $|(f-R)/R| \ll 1$, the extremum of the potential defines the time-dependent scalaron mass M_F

$$M_F^2 \equiv \frac{\partial^2 V_{eff}}{\partial F^2} = \frac{R}{3}\left(\frac{1}{m} - 1\right), \tag{2.9}$$

where $m = Rf_{,RR}/f_{,R}$ characterises the deviation of the background dynamics from the ΛCDM model ($m = 0$ at all times) [5]. Thus, viable $f(R)$ models are constructed such that the scalaron mass M_F is heavy enough in the regime of high matter density and becomes lighter at the present time to produce the accelerated expansion of the Universe. This process may be achieved via a chameleon mechanism [10, 34] ensuring that local gravity constraints are locally satisfied [31, 66]. On the other hand, using the quasi-static approximation [60] [1], the evolution of perturbations at linear order lead to expressions for the Newtonian potentials of the form:

$$\frac{k^2}{a^2}\Phi = -\frac{\kappa^2\delta\rho_m}{2F}\frac{2k^2/a^2 + 3M_F^2}{3k^2/a^2 + 3M_F^2}, \qquad \frac{k^2}{a^2}\Psi = -\frac{\kappa^2\delta\rho_m}{2F}\frac{4k^2/a^2 + 3M_F^2}{3k^2/a^2 + 3M_F^2}. \tag{2.10}$$

It is useful to introduce a function $A(a,k)$ [49] given by the squared ratio of the Compton wavelength to the physical wavelength of a mode:

$$A(a,k) = \left(\frac{k}{aM_F}\right)^2. \tag{2.11}$$

[1] A similar paper studies modified gravity theories without using the quasi-static approximation though [1].

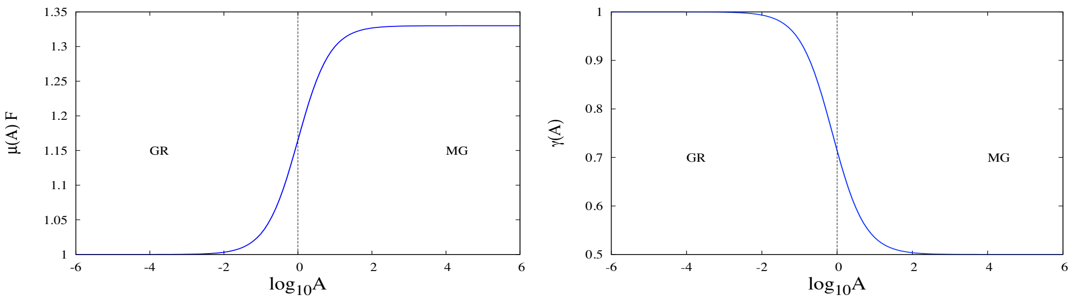

Figure 1. Functional behaviour of μ and γ. The vertical line $A(a, k) = 1$ represents the transition regime between GR and ST. Left to this line GR is recovered, whereas right to the line the growth of structures is enhanced by modifications of gravity.

Making a comparison of the equations for Φ and Ψ in (2.10) with those written in (1.3) - (1.4) for μ and γ, one has:

$$\mu(a,k)F = 1 + \frac{A(a,k)}{3+3A(a,k)}, \qquad \gamma(a,k) = 1 - \frac{2A(a,k)}{3+4A(a,k)}. \tag{2.12}$$

We observe that $f(R)$ models, through $\mu(a, k)$ and $\gamma(a, k)$, predict a characteristic scale-dependent growth of LSS which might be observationally detectable. The impact of the above relations on the evolution of the gravitational potentials and the growth of density perturbations is as follows: for a mode located in the *general relativistic regime* ($A \ll 1$) the scalaron behaves as a massive field making deviations from GR negligible, and the standard relation $\Phi \simeq \Psi$ is thus recovered. On the other hand, when a mode is situated within the *scalar-tensor regime* ($A \gg 1$), the scalaron behaves as a light particle, giving rise to an effective Newtonian constant $G_{\rm eff} = 4/(3F)$, and the relation between the metric potentials becomes $\Phi \simeq \Psi/2$. Therefore, the enhancement of the gravitational potential Ψ increases the growth rate of linear density perturbations on scales below the Compton wavelength [58]. If the transition between these two regimes ($A = 1$) occurred during matter domination, modifications of the observed matter power spectrum might signal deviations from the ΛCDM model [49]. The two regimes are described as follow:

$$\mu \simeq \frac{1}{F}, \qquad \gamma \simeq 1, \qquad A \ll 1 \qquad \text{GR}, \tag{2.13}$$

$$\mu \simeq \frac{4}{3F}, \qquad \gamma \simeq \frac{1}{2}, \qquad A \gg 1 \qquad \text{ST}. \tag{2.14}$$

Note that the factor F^{-1} corresponds to a rescaling of the Newtonian constant $G_{\rm N}$, for which the value is very close to unity for models that satisfy local and Galactic constraints. Figure 1 shows $\mu(A)$ and $\gamma(A)$ functions. The vertical axis represents the amplitude of μ and γ as a function of the squared ratio of the Compton wavelength to the physical wavelength, and the vertical line, $A(a, k) = 1$, the transition between the GR and the ST regime. Left of this line the behaviour is well described by (2.13), whereas on the right hand side, the observed enhancement of the growth of structures is described by (2.14).

2.3 A particular $f(R)$ model

We have, so far obtained expressions to describe the background evolution (2.6) and the cosmological perturbations (2.12) for a generic $f(R)$ model. Here, to exemplify our approach, we consider a particular $f(R)$ model and look at its observables. By construction, we assume $f(R)$ is a well-behaved function, continuous in all its derivatives. It also has to satisfy some further conditions in order to yield to a viable theory [6, 49, 58, 61]: $f_{,R} > 0$ to avoid the appearance of ghosts; $f_{,RR} > 0$ to avoid tachyonic instability; $f(R) \to R - 2\Lambda$ to include phenomenology of ΛCDM as a limiting case and recover BBN and CMB constraints at early times; $|F_0 - 1| \ll 1$ to satisfy Solar and Galactic constraints. Thus, we focus the study on a version of the Starobinsky model [58]:

$$f(R) = R - \lambda R_c \left[1 - \left(1 + \alpha \frac{R}{R_c} \right)^{-n} \right], \tag{2.15}$$

with positive constants λ, α and n, and R given by the solutions of equations (2.1) and (2.2). In the region of high density ($R \gg R_c$), model (2.15) and the Hu & Sawicki model [31] have a similar behaviour. Also model (2.15), with $n = 1$, closely mimics mCDTT [14] plus a cosmological constant, and the inverse squared-curvature model for $n = 2$ [42]. Some other $f(R)$ models with an exponential form [8] may also be considered as viable alternatives.
Given the $f(R)$ model (2.15), we are now able to compute its corresponding effective equation-of-state $w_{\rm eff}(z)$ (2.6), which dominates the dynamics of the late-time expansion rate, and μ, γ (2.12) to describe the perturbations. Another function to bear in mind is the rescaling factor F of the Newtonian constant, given by

$$F(R) - 1 = -\lambda \alpha n \left(1 + \alpha \frac{R}{R_c} \right)^{-n-1}. \tag{2.16}$$

An important point to emphasise is the behaviour presented by $|F - 1|$: as $R \gg R_c$, $|F - 1|$ becomes negligible, thus approaching the General Relativistic limit. Previous studies have chosen F_0 as a sampling parameter, although in our case, we consider it more natural to sample over α, with F_0 being a derived parameter. Also notice that at the extremum of the effective potential (2.8), the expansion history can approximate ΛCDM by setting $\lambda \simeq 6\Omega_{\rm eff,0}/\Omega_{m,0}$, with $\Omega_{m,0}$ the average matter density today and $\Omega_{\rm eff,0} = 1 - \Omega_{m,0}$.

Hence, for the model (2.15) the effective equation-of-state $w_{\rm eff}$ (2.6) and the squared ratio of the Compton wavelength to the physical wavelength of a mode $A(a, k)$ (2.11) are now parameterised in terms of α, n and the cosmological parameters $\Omega_{m,0}$ and H_0. These parameters together determine the epoch and scale on which modifications to GR may be relevant. In general, having a dependence on space and time makes the analysis more challenging. To understand the relationship of the new parameters with current observations let us consider some particular cases. In Figure 2 we show some of the relevant functions which parameterise the MG models using different α values, for $n = 1$ (left panel) and $n = 2$ (right panel); in both cases we have maintained fixed values of $\Omega_{m,0} = 0.25$ and $H_0 = 70$. The top panels show the behaviour of the normalised function $f(R)/(R - 2\Lambda)$. The middle panels display the effective equation-of-state for different values of α. We observe that at the background level, modified gravity models with $\alpha > 0.5$ ($n = 1$) or $\alpha > 0.2$ ($n = 2$) present deviations by just few percentages away from the cosmological constant. However, at the perturbations level (bottom panel), even larger values of α may be differentiated by the epoch they cross the regime line $A = 1$, and therefore μ be described by (2.14). A similar interpretation is used for the n parameter: a transition taking place at later times corresponds to higher values of n. One notices the existence of a pronounced degeneracy: for an increment in n, small values of

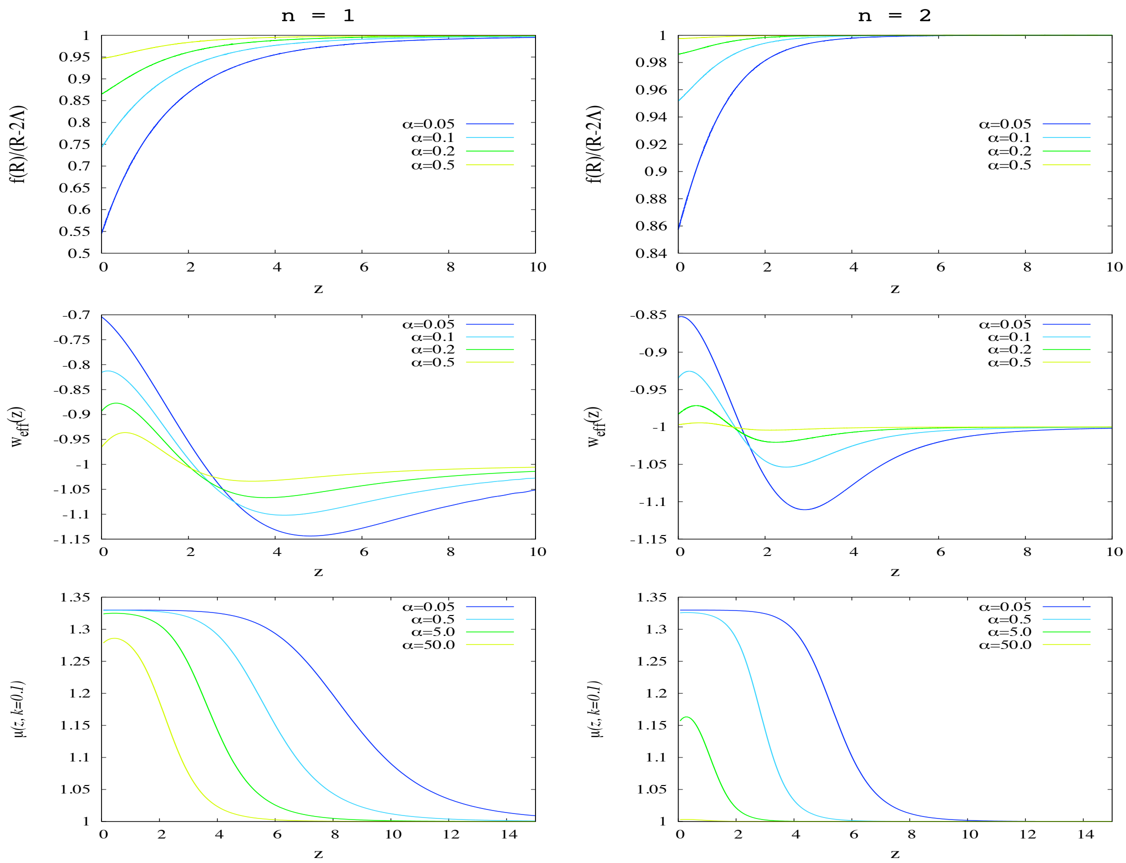

Figure 2. Redshift evolution of different functions used to describe a modified gravity model, with different values of α and n; at scale $k = 0.1$ Mpc^{-1}. $f(R)$ function normalised to the standard ΛCDM (top), effective equation-of-state w_{eff} (middle) and the screen mass function (bottom).

α mimic the same behaviour, for instance, models with $\{n = 1, \alpha = 5\}$ and $\{n = 2, \alpha = 0.5\}$ behaved similarly. Finally, we emphasise that the screened mass function μ and the gravitational slip parameter γ must equal one, at high redshifts to recover GR. That is, in order to maintain the properties of Big Bang Nucleosynthesis (BBN) at early times, we should impose the condition

$$\mu(z, k) = \gamma(z, k) = 1, \qquad\qquad z \gtrsim 30. \tag{2.17}$$

This condition is translated into $A \ll 1$ at $z \gtrsim 30$ for the whole range of physically relevant wavenumbers.

3 Analysis

We seek to impose constraints on the aforementioned parameterisations from cosmological observables as well as determine which model best describes current data. Since the evolution of matter perturbations and gravitational potentials may differ from standard GR, observations of CMB anisotropy, cosmic evolution and growth of structure are important probes for discriminating amongst modified gravity models. In order to compare the modified gravity influence on observable quantities, such as the CMB, matter power spectrum and luminosity distances, we incorporate μ and γ functions to the standard Boltzmann CAMB code [38] up to $z \sim 30$, when deviations introduced by modified gravity become negligible. Different versions of the Modified CAMB code have also been released by e.g. [30, 32, 68]. Also, with the use of a post-Friedmann prescription (PPF) implemented by [22], we have included the effective equation-of-state $w_{\rm eff}$ predicted by a modified gravity model.

The combination of current observations at different epochs and scales will strengthen the constraints on the parameters allowing one to discriminate amongst theories. To constrain the parameter-space in each model, we include temperature and polarisation measurements of the Wilkinson Microwave Anisotropy Probe 7-year (WMAP7; [35]), together with the Atacama Cosmology Telescope (ACT; [19]), the Quest (Q and U Extra-Galactic Sub-mm Telescope) at DASI (Degree Angular Scale Interferometer) (QUAD; [11]), and the Background Imaging of Cosmic Extragalactic Polarization (BICEP; [15]). In addition to CMB data, we incorporate distance measurements of 557 Supernovae Ia from the Supernova Cosmology Project Union 2 compilation (SCP; [4]). We also consider baryon density information from Big Bang Nucleosyntesis (BBN; [12]), and impose a Gaussian prior on H_0 using measurements from the Hubble Space Telescope key project (HST; [53]). Additional to CMB and SNe observations, we include large scale structure data from the Sloan Digital Sky Survey (SDSS) Data Release 7 (DR7) Luminous Red Galaxy (LRG) power spectrum [51].

Throughout the analysis, we consider purely Gaussian adiabatic scalar perturbations, neglecting contributions from tensors and massive neutrinos as dark matter. We base our analysis on a modified ΛCDM model which assumes a FRW background universe specified by the following parameters: the physical baryon density $\Omega_{\rm b}h^2$ and cold dark matter density $\Omega_{\rm DM}h^2$, respectively, relative to the critical density (h is the dimensionless Hubble parameter such that $H_0 = 100h$ kms^{-1}Mpc^{-1}), the ratio θ of the sound horizon to angular diameter distance at last scattering, the optical depth τ at reionisation, the amplitude $A_{\rm s}$ and spectral index $n_{\rm s}$ of the primordial power spectrum, the running of the index $n_{\rm run} \equiv dn_{\rm s}/d\ln k$ and the primordial curvature perturbation amplitude $A_{\rm s}$, defined at pivot scale $k_0 = 0.015$ Mpc^{-1}. Aside from the Sunyaev-Zel'dovich (SZ) amplitude A_{SZ}, the ACT likelihood incorporates two additional secondary parameters: the Poisson point source power A_p and the clustered point source power A_c. The parameterisations of MG models contain the ΛCDM model as a subset in their parameter space, thus the flat priors on the parameters in common are kept identical for each considered case: $\Omega_{\rm b,0}h^2 = [0.01, 0.03]$, $\Omega_{\rm DM,0}h^2 = [0.05, 0.2]$, $\theta = 100 \times [1, 1.1]$, $\tau = [0.01, 0.3]$, $\ln[10^{10}A_{\rm s}] = [2.5, 4]$, $n_{\rm s} = [0.5, 1.2]$ and $n_{\rm run} = [-0.1, 0.1]$. With regards to the additional parameters, we assume three illustrative cases: $n = 1, 2$ with flat priors $\alpha = [0, 5]$ and $\alpha = [0, 50]$ respectively; and varying n within the range $n = [0, 2]$ and $\alpha = [0, 50]$.

The preferred model given current cosmological observations is selected through its Bayes factor. In general, the calculation of the Bayesian evidence is a very computationally demanding process. To carry out the model selection we incorporate into the CosmoMC software [37] a substantially improved and fully-parallelized version of the *nested sampling* algorithm called MultiNest [24, 25]. The Bayes factor $\mathcal{B}_{ij}$, or equivalently the difference in log evidences $\ln \mathcal{Z}_i - \ln \mathcal{Z}_j$, provides a measure of how well model i fits the data compared to model j. A suitable guideline for making qualitative conclusions has been laid out by Jeffreys: if $\mathcal{B}_{ij} < 1$ model i fits equally well as model j, $1 < \mathcal{B}_{ij} < 2.5$ constitutes significant evidence, $2.5 < \mathcal{B}_{ij} < 5$ is strong evidence, while $\mathcal{B}_{ij} > 5$ would be considered decisive [59, 64].

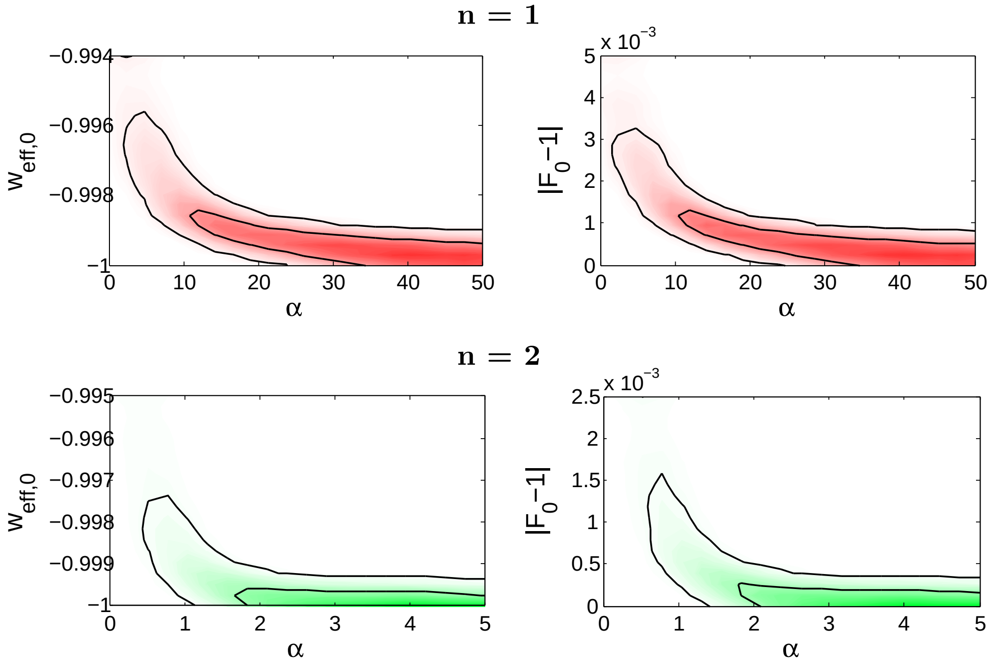

Figure 3. 2D Marginalised posterior distributions of sampling parameter α along with derived parameters: the effective equation-of-state $w_{\rm eff,0}$ and the field amplitude $|F_0 - 1|$ at present time; using $n = 1$ (top panels) and $n = 2$ (bottom panels).

4 Results

In this section we present the resulting posterior distributions and model evidences computed from the gravity models using our data sets. Despite the additional parameters, the mean values of the standard cosmological parameters remained basically unaffected. That is, their likelihoods peak around standard ΛCDM values, and so we do not consider them further.

We have restricted our analysis to the model presented in (2.15). As we pointed out, this particular parameterisation can be studied in terms of α, n and the cosmological parameters $\Omega_{m,0}$, H_0; although, constraints on H_0 and $\Omega_{m,0}$ present no significant deviations from the ΛCDM model. Figure 3 shows 2-D marginalised posterior distributions of the modified gravity parameter α along with the rescaling factor and the effective equation of state at the present time, $|F_0 - 1|$ and $w_{\rm eff,0}$, respectively. Top panel corresponds to $n = 1$, whereas $n = 2$ to bottom panel. We notice that the effective equation-of-state at the present time $w_{\rm eff,0} \equiv w_{\rm eff}(z = 0)$ exhibits only slight deviations from the cosmological constant with $-1 < w_{\rm eff,0} < -0.996$ ($n = 1$) and $-1 < w_{\rm eff,0} < -0.997$ ($n = 2$). At the perturbations level, preferred values of the field amplitude at the present time are $|F_0 - 1| < 0.003$ ($n = 1$), and $|F_0 - 1| < 0.002$ ($n = 2$). Higher values of α may be seen as approaching the ΛCDM model where $|F_0 - 1| \to 0$ and $w_{\rm eff,0} \to -1$, as illustrated previously in Figure 2. Modified gravity models lead to broader constraints on density fluctuations, σ_8, where higher values are preferred, as shown in Table 1. With regards to the case where n is treated as a free parameter, $0 < n < 2$, we observe that constraints on α, $w_{\rm eff,0}$ and $|F_0 - 1|$ are slightly broadener: $-1 < w_{\rm eff,0} < -0.99$ and $|F_0 - 1| < 0.007$ (see Figure 4). Here it is noticeable that higher values of n lead to ΛCDM standard values, for instance, on the behaviour of σ_8 shown in the right panel of Figure 4.

$0 < n < 2$

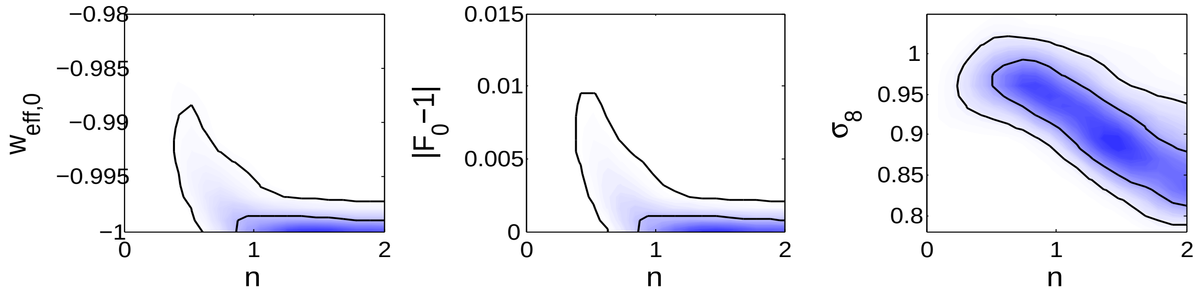

Figure 4. 2D Marginalised posterior distributions of sampling parameter α along with the effective equation-of-state $w_{\rm eff,0}$, the field amplitude at the present epoch $|F_0 - 1|$, and the density fluctuations σ_8 in spheres of radius $R = 8h^{-1}$ Mpc.

Table 1. Constraints on modified-gravity parameters. For one-tailed distributions the upper limit 95% CL is given. For two-tailed the 68 % is shown.

	ΛCDM	$n=1$	$n=2$	$0<n<2$
α	–	> 0.7	> 0.4	unconstrained
n	–	–	–	> 0.53
λ	–	15.8 ± 1.3	15.6 ± 1.3	15.4 ± 1.4
H_0	68.5 ± 1.4	69.7 ± 1.4	69.6 ± 1.4	69.4 ± 1.4
Ω_m	0.293 ± 0.017	0.276 ± 0.017	0.278 ± 0.016	0.281 ± 0.018
σ_8	0.819 ± 0.019	0.945 ± 0.027	0.914 ± 0.032	0.915 ± 0.049
$w_{\rm eff,0}$	–	< -0.996	< -0.998	< -0.99
$\lvert F_0 - 1\rvert$	–	< 0.003	< 0.002	< 0.007
$-2\Delta \ln \mathcal{L}_{\rm max}$	–	-0.22	-1.21	-1.49
$\mathcal{B}_{i,\Lambda}$	–	$+0.5 \pm 0.3$	$+1.0 \pm 0.3$	$+0.8 \pm 0.3$

The summary of the parameter constraints is given in Table 1. One-tailed constraints are quoted at 95% C.L. whilst for two tails 68% is shown. We note that similar constraints on λ and $|F_0 - 1|$ have been found by using a subset of the parameter-space and particular values of the wavenumber k [55, 67].

We have computed the Bayesian evidence for each model to perform a model comparison, according to the Jeffreys guideline. When the set of models are ranked with respect to its Bayesian value $f(R)$ models are preferred despite having extra parameters, when compared to the ΛCDM model. Important attention is paid to the evidence of the $f(R)$ with $n = 2$, which is significantly preferred, $\mathcal{B}_{i,\Lambda} = +1.0 \pm 0.3$, and over the rest of the models, shown in the last row of Table 1.

5 Conclusions

We have undertaken an analysis of modified gravity models by studying its background history as well as linear perturbations. At the background level, the dynamics of $f(R)$ models is encoded in the effective equation-of-state $w_{\rm eff}$, whereas at the perturbations level it depends on the ratio of the metric potentials μ and the effective gravitational constant γ. Initially, we provided a description for these three functions with a general $f(R)$ model. Then, we use as an example a variant of the Starobinsky model to constrain the parameter-space using current cosmological observations. We have found that constraints on the base parameters are largely unaffected by the introduction of these three effective functions. That is, best-fit values for the standard parameters shift by less than 1σ. The only notable exception is σ_8, whose marginalised uncertainties increase by up to a factor of two upon the introduction of extra parameters. This is consistent with the observation that μ and γ principally modify the growth history of cosmological perturbations. Figure 5 shows the reconstructed $f(R)$, $w_{\rm eff}$ and μ at $k = 0.1$ Mpc^{-1}, using posterior samples in the region (2σ) around best-fit values. We observe that measurements on the screen mass function present slight deviations from unity at the latest times ($z < 2$), but still consistent with values $\mu = \gamma = 1$. Deviations from $\mu = 1$ increase at smaller scales (larger k), however at the smallest scales non-linear physics plays an important role and linear perturbation theory is no longer valid. Similarly for the $f(R)$ function. Larger differences, between $f(R)$ and $R - 2\Lambda$, exist at the present time (less than 1%), whereas at early epochs $f(R)$ approaches the standard ΛCDM model. Finally, we have applied a Bayesian criterion to carry out cosmological model selection and found, in the sense of Jeffreys guideline, that the variant of the Starobinsky $f(R)$ model with $n = 2$ provides a slightly better fit compared to the ΛCDM model: $\mathcal{B}_{i,\Lambda} = 1.0 \pm 0.3$. Although a particular $f(R)$ was chosen, this analysis may be easily extended to a broad classes of models, with the use of high accurate experiments, in order to look for deviations from the cosmological constant model.

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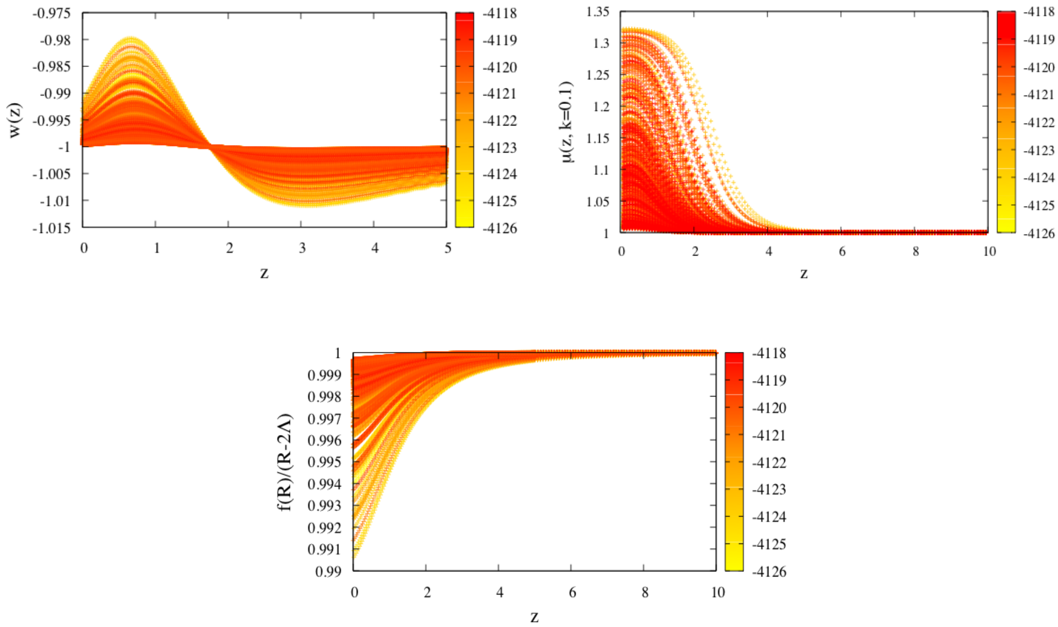

Figure 5. Reconstruction of the effective equation-of-state $w_{\rm eff}$, the screen mass function μ and the normalised $f(R)$ function, using best-fit values obtained in the analysis, for a $f(R)$ with $n = 2$. The colour-code indicates the ln(Likelihood), where darker regions represent an improved fit.

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